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GSM/M-22

1613

SPECIAL FUNCTIONS AND INTEGRAL TRANSFORMS

Paper–BM-242

Time Allowed : 3 Hours] [Maximum Marks : 40

Note : Attempt **five** questions in all, selecting **one** question from each Unit. Question No. **1** is compulsory. All questions carry equal marks.

Compulsory Question

1. (a) Prove that $J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$.

(b) Show that : $\int_{-1}^{1} P_n(x)\,dx = 2$ if $n = 0$.

(c) Find the Laplace Transform of cosh at sin at.

(d) Find the Fourier transform of $f(x) = e^{-a|x|}$, where $a > 0$ and $x \in (-\infty, \infty)$. 2×4=8

UNIT–I

2. (a) Solve $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = e^{-x}$ where $y(0) = 2$ and $y'(0) = -3$. 4

(b) Show that $J_{-n}(x) = (-1)^n J_n(x)$ where n is any integer. 4

3. (a) Find the solution of $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0$ in terms of Bessel's function. 4

(b) Verify that the Bessel's function $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ satisfies the Bessel's equation of order $-\frac{1}{2}$. 4

UNIT–II

4. (a) Show that $(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x), |x| \le 1, t < 1$ where $P_n(x)$ is Legendre's function of order n. 4

(b) Prove that :

$$\int_{-1}^{1} x^2 P_n^2(x)\,dx = \frac{1}{8(2n-1)} + \frac{2}{4(2n+1)} + \frac{1}{8(2n+3)}.$$ 4

5. (a) Show that $H_n(x) = 2^n\left[\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right]x^n$. 4

(b) Prove that :

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}[H_n(x)]^2\,dx = \sqrt{\pi}\,2^n n!\left(n + \frac{1}{2}\right).$$ 4

UNIT–III

6. (a) Evaluate $\int_0^\infty t\, e^{-t} \sin^4 t\, dt$. 4

(b) Find $L^{-1}\left[\log \frac{s^2+1}{(s-1)^2}\right]$. 4

7. (a) Using convolution theorem evaluate $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$. 4

(b) Solve the equation by transform method :

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=e^{-t} \text{ where } y(0)=y'(0)=1.$$ 4

UNIT–IV

8. (a) Find the Fourier transform of f(x)

$$\text{if } f(x)=\begin{cases}\frac{1}{2\epsilon}, & |x|\le\epsilon \\ 0, & |x|>\epsilon\end{cases}.$$ 4

(b) Using Parseval's identity, Prove :

$$\int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}.$$ 4

9. (a) Using the Fourier sine transform, solve the partial differential equation $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$ with the boundary condition.

(i) $u=u_0$ when $x=0, t>0$ and the initial condition.

(ii) $u=0$ when $t=0, x>0$. 4

(b) Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$, given that :

(i) $u(0, t)=0$ (ii) $u(\pi, t)=0$ (iii) $u(x, 0)=2x$

when $0<x<\pi$ $t>0$. 4