## GSE/M-20

MATHEMATICS
(Number Theory and Trigonometry)
Paper: BM-121
Time : Three Hours]
[Maximum Marks : 40
Note : Attempt five questions in all. Question No. 1 is compulsory. Select one question from each section.

## Compulsory Question

1. (a) If $a \mid b c$ and $(a, b)=1$, then prove that $a \mid c$.
(b) Find the least positive integer $(\bmod 11)$ to which 282 is congruent.
(c) Find all possible values of $n$ which satisfies $\phi(n)=23$.
(d) Express $\cos ^{6} \theta$ in terms of cosines of multiples of $\theta$.

1
(e) Prove that $\tan ^{-1}\left(\frac{x}{\sqrt{a^{2}-x^{2}}}\right)=\sin ^{-1}\left(\frac{x}{a}\right)$.

## SECTION-I

2. (a) Prove that there are infinitely many pairs of integers $x$, $y$ satisfying $x+y=100$ and $(x, y)=10$ simultaneously.
(b) Solve the congruence $342 x \equiv 5(\bmod 13)$.
3. (a) State and prove Wilson's theorem. 4
(b) Show that $n^{16}-a^{16}$ is divisible by 85 if $n$ and $a$ are coprime to 85 .

## SECTION-II

4. (a) Find all integers that give the remainder $1,3,2$ when divided by $4,5,7$ respectively. 4
(b) Prove that $\phi(n)=\phi(n+2)$ is satisfied by $n=2(2 p-1)$ whenever $p$ and $2 p-1$ are both odd prime.
5. (a) Find the highest power of 180 in 102!. 4
(b) Evaluate $\left(-\frac{168}{11}\right)$.

4

## SECTION-III

6. (a) If $(1+x)^{n}=p_{0}+p_{1} x+p_{2} x^{2}+\ldots$, show that
(i) $p_{1}-p_{3}+p_{5}+\ldots \ldots=2^{n / 2} \sin \frac{n \pi}{4}$.
(ii) $p_{0}-p_{2}+p_{4}+\ldots \ldots=2^{n / 2} \cos \frac{n \pi}{4}$.
(b) Show that

$$
\begin{equation*}
\tan \frac{\theta}{7}+\tan \frac{\theta+\pi}{7}+\cdots+\tan \frac{\theta+6 \pi}{7}=7 \tan \theta . \tag{4}
\end{equation*}
$$

7. (a) Express $\sin ^{6} \theta \cos ^{2} \theta$ in a series of cosines of multiples of $\theta$. 4
(b) Separate tanh $(x+i y)$ into real and imaginary parts. 4

## SECTION-IV

8. (a) Prove that principal value of $\frac{(a+i b)^{p+i q}}{(a-i b)^{p-i q}}$ is

$$
\cos 2(p \alpha+q \log r)+i \sin 2(p \alpha+q \log r),
$$

where $r=\sqrt{a^{2}+b^{2}}$ and $\alpha=\tan ^{-1} \frac{b}{a}$.
(b) Prove that

$$
\sin ^{-1}(\operatorname{cosec} \theta)=\left[2 n+(-1)^{n}\right] \frac{\pi}{2}+i(-1)^{n} \log \cot \frac{\theta}{2} \cdot 4
$$

9. (a) Show that

$$
\frac{\pi}{4}=\frac{17}{21}-\frac{713}{81.343}+\cdots+\frac{(-1)^{n+1}}{2 n-1}\left[\frac{2}{3} \cdot 9^{1-n}+7^{1-2 n}\right]+\cdots
$$

(b) Sum the series

$$
\tan ^{-1} \frac{1}{3}+\tan ^{-1} \frac{1}{7}+\tan ^{-1} \frac{1}{13}+\cdots+\text { to } n \text { terms }
$$

and deduce the sum to infinity.

